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A remark on the separation by immersions in codimension 1

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Abstract

Let $f: M^{n-1} \to N^n$ be an immersion with normal crossings from a compact connected $(n-1)$-manifold M into a connected, open or compact n-manifold N, where M and N can have boundaries. In this paper, we give a necessary and sufficient condition for f to be an embedding using the number of connected components of $N - f(M)$. We also obtain an estimate from the above for the number of connected components of $N - f(M)$ for f with only double points as its self-intersection points.

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1. Introduction

Let M^{n-1} and N^n be *closed* connected manifolds of dimensions $n-1$ and n respectively and $f: M \to N$ an immersion with normal crossings. In [2] a characterization of f as an embedding is obtained, which is a converse to the Jordan–Brouwer theorem, under certain homological conditions. Note that the same characterization had been obtained in [1] under a more restricted condition.

In this paper, we give an extension of the results in [2] to the cases where M and N can have boundaries and N can be open. We also give an estimate from the above for the number of connected components of the complement of $f(M)$ for

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the extended cases. The techniques used for the proofs are essentially the same as those used in [1,2] although ours are rather sophisticated.

In order to deal with maps between manifolds with boundary, we need the following definition.

Definition 1.1. Let $f : M \to N$ be a smooth map between manifolds with boundary. We say that f is *neat* if $f^{-1}(\partial N) = \partial M$ and f is transverse to ∂N. Furthermore, a neat map f is an *immersion with normal crossings* if $f \mid \mathrm{Int}\ M$ and $f \mid \partial M : \partial M \to \partial N$ are both immersions with normal crossings in the usual sense.

Our main results of this paper are as follows.

Theorem 1.2. *Let $f : M \to N$ be an immersion with normal crossings between connected manifolds with boundary such that M is an $(n-1)$-dimensional compact manifold and that N is an n-dimensional manifold. Suppose that the self-intersection set of f is nonempty. In the following, β_0 denotes the number of connected components.*

(1) *If M is orientable and $H_1(N; \mathbb{Z}_2) = 0$, then $\beta_0(N - f(M)) \geqslant 3$.*

(2) *If $H_1(M; \mathbb{Z}_2) = 0$ and $H_{n-1}(N, \partial N; \mathbb{Z}_2) = 0$, then $\beta_0(N - f(M)) \geqslant 3$.*

(3) *If M and N are orientable and $H_{n-1}(N, \partial N; \mathbb{Z}_2) = 0$, then $\beta_0(N - f(M)) \geqslant 3$.*

(4) *If $i_* : H_{n-1}(f(M), f(\partial M); \mathbb{Z}_2) \to H_{n-1}(N, \partial N; \mathbb{Z}_2)$ vanishes and if the normal bundle of the immersion f is trivial, then $\beta_0(N - f(M)) \geqslant 3$, where $i : f(M) \to N$ is the inclusion map.*

Theorem 1.3. *Let $f : M \to N$ be an immersion with normal crossings between connected manifolds with boundary such that M is a compact $(n-1)$-dimensional manifold and that N is an n-dimensional manifold with $H_1(N; \mathbb{Z}_2) = 0$ or $H_{n-1}(N, \partial N; \mathbb{Z}_2) = 0$. If f has only double points as its self-intersection points, then $\beta_0(N - f(M)) \leqslant 2 + \beta_0(B)$, where $B = f(A)$ and A is the self-intersection set of f. Furthermore, if $(A, A \cap \partial M)$ is null-homologous in $(M, \partial M)$, then the equality holds.*

Note that both of the above theorems are valid even if ∂M or ∂N is empty. The estimate corresponding to Theorem 1.3 had been obtained by Izumiya and Marar [4] when $n = 3$ and M has no boundary. We also note that Theorem 1.3 has recently been obtained independently by Nuño Ballesteros [5] when M has no boundary.

2. Proofs of theorems

In all that follows, M^{n-1} and N^n are connected manifolds with boundary of dimensions $n-1$ and n respectively, M is compact, $f : M \to N$ is an immersion with normal crossings in the sense of Definition 1.1, and the homologies are with coefficients in $\mathbb{Z}_2$. Note that N may be an open manifold. Let $A(\subset M)$ be the self-intersection set of f; i.e., $A=\{p \in M: f^{-1}(f(p)) \neq \{p\}\}$. We denote by $B = f(A)$ the set of multiple values of f. Furthermore, set $\partial A = A \cap \partial M$ and $\partial B = f(\partial A) = B \cap \partial N$. Note that ∂A coincides with the self-intersection set of the immersion with normal crossings $f \mid \partial M : \partial M \to \partial N$.

Lemma 2.1. *If $A \neq \emptyset$, then it carries a nonzero* mod 2 *fundamental class* $[A, \partial A] \in H_{n-2}(A, \partial A)$.

Lemma 2.1 can be proved by a method similar to the proof of [3, Lemma 2.3] with $m=2$. Note that the map $f^{(2)}: M^{(2)} \to N^{(2)}$ — the 2-fold product map induced by f — is a smooth map between manifolds with corners; however, we can apply the operation of smoothing the corners and then $f^{(2)}$ becomes a neat map such that $f^{(2)} \mid \mathrm{Int}\ M^{(2)}$ (respectively $f^{(2)} \mid \partial M^{(2)}$) is transverse to the diagonal Δ_N (respectively $\partial \Delta_N = \Delta_{\partial N}$) off the diagonal Δ_M. Then the same argument as in [3] can apply.

Lemma 2.2. *If $H_1(N)=0$, then $\beta_0(N-f(M)) = 1 + \beta_{n-1}(f(M), f(\partial M))$, where $\beta_i(X, Y) = \dim H_i(X, Y)$ for a pair of topological spaces (X, Y).*

Proof. Consider the exact sequence:

$$\tilde{H}^0(N) \to \tilde{H}^0(N-f(M)) \to H^1(N, N-f(M)) \to H^1(N).$$

Since $\tilde{H}^0(N) = H^1(N) = 0$, we have $\beta_0(N-f(M)) = 1 + \dim\ H^1(N, N-f(M))$. On the other hand, by excision and the Poincaré–Lefschetz duality, we see that $H^1(N, N-f(M)) \cong H_{n-1}(f(M), f(\partial M))$. $\square$

Lemma 2.3. (1) $\beta_0(N-f(M)) \geqslant 1 + \dim\ \ker(i_*: H_{n-1}(f(M), f(\partial M)) \to H_{n-1}(N, \partial N))$, *where $i: f(M) \to N$ is the inclusion map.*

(2) $\beta_{n-1}(f(M), f(\partial M)) = 1 + \dim\ \ker(\alpha: H_{n-2}(A, \partial A) \to H_{n-2}(B, \partial B) \oplus H_{n-2}(M, \partial M))$, *where $\alpha = (f \mid A)_* \oplus j_*$ and $j:(A, \partial A) \to (M, \partial M)$ is the inclusion map.*

For the proof of Lemma 2.3, first we prove the following.

Lemma 2.4. *If N is compact, we have $\beta_0(N-f(M)) = 1 + \dim \ker (i_*: H_{n-1}(f(M), f(\partial M)) \to H_{n-1}(N, \partial N))$.*

Proof. Consider the following exact sequence of the triad $(N, \partial N \cup f(M), \partial N)$:

$$H_n(\partial N \cup f(M), \partial N) \to H_n(N, \partial N) \to H_n(N, \partial N \cup f(M))$$
$$\to H_{n-1}(\partial N \cup f(M), \partial N) \xrightarrow{i_*} H_{n-1}(N, \partial N).$$

Note that $H_n(\partial N \cup f(M), \partial N) = 0$, since $\partial N \cup f(M)$ is an $(n-1)$-dimensional polyhedron. Furthermore, since N is compact, we have $H_n(N, \partial N) \cong \mathbb{Z}_2$. On the other hand, by the Poincaré–Lefschetz duality, we have $H^0(N-f(M)) \cong H_n(N, \partial N \cup f(M))$. Furthermore, we have $H_{n-1}(\partial N \cup f(M), \partial N) \cong H_{n-1}(f(M), f(\partial M))$ by excision. Hence we have the required equality. □

Proof of Lemma 2.3. (1) Set $K = \ker i_* \subset H_{n-1}(f(M), f(\partial M))$. Since $f(M)$ is a compact polyhedron, there exist a finite number of generators $\mu_k = [c_k]$ $(1 \leqslant k \leqslant m)$ of K, where c_k are cycles in $(f(M), f(\partial M))$. Since $i_*([c_k]) = 0$, there exist n-chains C_k $(1 \leqslant k \leqslant m)$ in N such that $\partial C_k - c_k$ is a chain in ∂N. Take a connected compact codimension-0 submanifold W of N with corners such that $f(M) \cup \bigcup_{k=1}^m C_k \subset W - \partial_+ W$, where $\partial_+ W = \overline{(N-W)} \cap W$. Such a submanifold always exists. Note that $\beta_0(N-f(M)) \leqslant \beta_0(W-f(M))$ in general. However, since $\beta_0(N-f(M))$ and $\beta_0(W-f(M))$ are finite, we may assume that $\beta_0(N-f(M)) = \beta_0(W-f(M))$, adding some embedded 1-handles to W if necessary. Then it is easy to see that $K = \ker(i_* : H_{n-1}(f(M), f(\partial M)) \to H_{n-1}(W, \partial_- W))$, where $\partial_- W = \overline{\partial W - \partial_+ W}$.

On the other hand, consider the double N′ of W, which is the union of two copies of W attached along $\partial_+ W$. Note that W is a compact n-dimensional manifold with boundary the double of $\partial_- W$. Since W is a retract of N', we see that $i'_* : H_{n-1}(W, \partial_- W) \to H_{n-1}(N', \partial N')$ is injective, where $i' : (W, \partial_- W) \to (N', \partial N')$ is the canonical inclusion. Therefore, $K = \ker(\bar{i}_* : H_{n-1}(f(M), f(\partial M)) \to H_{n-1}(N', \partial N'))$, where $\bar{i}$ is the composite of the inclusion maps $(f(M), f(\partial M)) \subset (W, \partial_- W) \subset (N', \partial N')$. Furthermore, it is easy to see that $\beta_0(N-f(M)) = \beta_0(W-f(M)) \geqslant \beta_0(N' - \bar{i}(f(M)))$.

Then, since N' is compact, we see that $\beta_0(N-f(M)) \geqslant \beta_0(N' - \bar{i}(f(M))) = 1 + \dim K$ by Lemma 2.4. This completes the proof of (1).

(2) Consider the following commutative diagram with exact rows:

$$\begin{array}{ccc}
\longrightarrow H_i(\partial M \cup A, \partial M) & \longrightarrow & H_i(M, \partial M) \\
\downarrow f_* & & \downarrow f_* \\
\longrightarrow H_i(f(\partial M \cup A), f(\partial M)) & \longrightarrow & H_i(f(\mathrm{M}), f(\partial M))
\end{array}$$

$$\begin{array}{ccc}
\longrightarrow H_i(M, \partial M \cup A) & \longrightarrow & H_{i-1}(\partial M \cup A, \partial M) \\
\downarrow f_* \cong & & \downarrow f_* \\
\longrightarrow H_i(f(M), f(\partial M \cup A)) & \longrightarrow & H_{i-1}(f(\partial M \cup A), f(\partial M))
\end{array}$$

$$\begin{array}{c}
\longrightarrow H_{i-1}(M, \partial M) \longrightarrow \\
\downarrow f_* \\
\longrightarrow H_{i-1}(f(M), f(\partial M)) \longrightarrow .
\end{array}$$

Note that, by excision and the Poincaré–Lefschetz duality, $f_* : H_i(M, \partial M \cup A) \to$

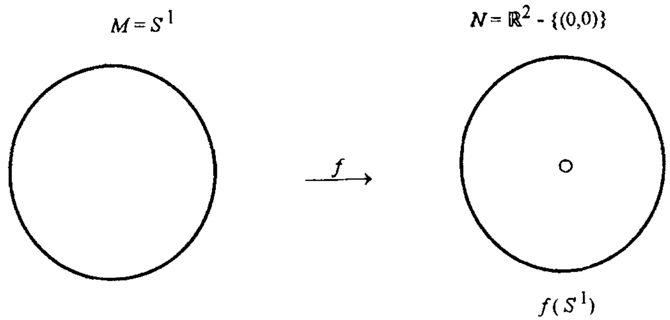


Fig. 1.

$H_i(f(M),\ f(\partial M \cup A))$ is an isomorphism. Hence, we have the following exact sequence:

$$\begin{aligned} &H_{n-1}(\partial M \cup A,\ \partial M) \longrightarrow H_{n-1}(M,\ \partial M) \oplus H_{n-1}(f(\partial M \cup A),\ f(\partial M)) \\ &\longrightarrow H_{n-1}(f(M),\ f(\partial M)) \longrightarrow H_{n-2}(\partial M \cup A,\ \partial M) \\ &\xrightarrow{\alpha} H_{n-2}(M,\ \partial M) \oplus H_{n-2}(f(\partial M \cup A),\ f(\partial M)). \end{aligned}$$

Note that $H_{n-1}(\partial M \cup A,\ \partial M) = 0$, $H_{n-1}(M,\ \partial M) \cong \mathbb{Z}_2$, $H_{n-1}(f(\partial M \cup A),\ f(\partial M)) = 0$, $H_{n-2}(\partial M \cup A,\ \partial M) \cong H_{n-2}(A,\ \partial A)$ and $H_{n-2}(f(\partial M \cup A),\ f(\partial M)) \cong H_{n-2}(B,\ \partial B)$. Hence, we have the required equality. □

Note that, in Lemma 2.3(1), the equality does not hold in general (see Fig. 1, where $2 = \beta_0(N - f(M)) > 1 + \dim \ker i_* = 1$). However, in special cases, we have the equality as follows.

Lemma 2.5. *If* $i_* : H_{n-1}(f(M),\ f(\partial M)) \to H_{n-1}(N,\ \partial N)$ *vanishes, then we have* $\beta_0(N - f(M)) = 1 + \beta_{n-1}(f(M),\ f(\partial M))$.

Proof. Let V be a regular neighborhood of $f(M)$ in N and set $E = \overline{N - V}$. Consider the following exact sequence of the triad $(N,\ E \cup \partial N \cup f(M),\ E \cup \partial N)$:

$$\begin{aligned} &H_n(E \cup \partial N \cup f(M),\ E \cup \partial N) \to H_n(N,\ E \cup \partial N) \to H_n(N,\ E \cup \partial N \cup f(M)) \\ &\longrightarrow H_{n-1}(E \cup \partial N \cup f(M),\ E \cup \partial N) \xrightarrow{i_*} H_{n-1}(N,\ \partial E \cup \partial N). \end{aligned}$$

Note that $H_n(E \cup \partial N \cup f(M),\ E \cup \partial N) = 0$, $H_n(N,\ E \cup \partial N) \cong \mathbb{Z}_2$, $H_n(N,\ E \cup \partial N \cup f(M)) \cong H^0(V - f(M))$, $H_{n-1}(E \cup \partial N \cup f(M),\ E \cup \partial N) \cong H_{n-1}(f(M),\ f(\partial M))$ and that i_* is the zero map. Hence we have $\beta_0(N - f(M)) \leqslant \beta_0(V - f(M)) = 1 + \beta_{n-1}(f(M),\ f(\partial M))$. Combining this with Lemma 2.3(1), we have the required equality. □

Remark 2.6. Suppose that $i_* : H_{n-1}(f(M),\ f(\partial M)) \to H_{n-1}(N,\ \partial N)$ vanishes. Then by the proof of Lemma 2.5, we see that $\beta_0(N - f(M)) = \beta_0(V - f(M))$. On the other hand, $\beta_0(N-f(M)) = 1 + \dim H_{n-1}(f(M), f(\partial M))$ by Lemma 2.5 and $\beta_0(V-f(M)) = 1 + \dim \ker(i_*: H_{n-1}(f(M), f(\partial M)) \to H_{n-1}(V, \partial V))$ by Lemma 2.4. Hence the map $i_*: H_{n-1}(f(M), f(\partial M)) \to H_{n-1}(V, \partial V)$ necessarily vanishes.

We can prove the following lemma by the same argument as in [2].

Lemma 2.7. *Assume that the normal bundle of the immersion f is trivial and that $f_*([M, \partial M]) = 0$ in $H_{n-1}(N, \partial N)$ or $f_*: H_1(M) \to H_1(N)$ vanishes. Then $j_*([A, \partial A]) = 0$ in $H_{n-2}(M, \partial M)$.*

Now we proceed to the proofs of the theorems.

Proof of Theorem 1.2. (1) Since $H_1(N) = 0$, we see that $\beta_0(N-f(M)) = 1 + \beta_{n-1}(f(M), f(\partial M))$ by Lemma 2.2. Furthermore, $\beta_{n-1}(f(M), f(\partial M)) = 1 + \dim \ker \alpha$ by Lemma 2.3(2). On the other hand, since M is orientable and $H_1(N) = 0$, the normal bundle of the immersion f is trivial (see [2, Lemma 2.1]). Hence we have $j_*([A, \partial A]) = 0$ by Lemma 2.7. Thus $[A, \partial A] \in H_{n-2}(M, \partial M)$ satisfies $[A, \partial A] \neq 0$ and $\alpha([A, \partial A]) = 0$ (see also [2, Lemma 2.2]), and hence we see that $\beta_{n-1}(f(M), f(\partial M)) \geqslant 2$. Thus we have $\beta_0(N-f(M)) \geqslant 3$.

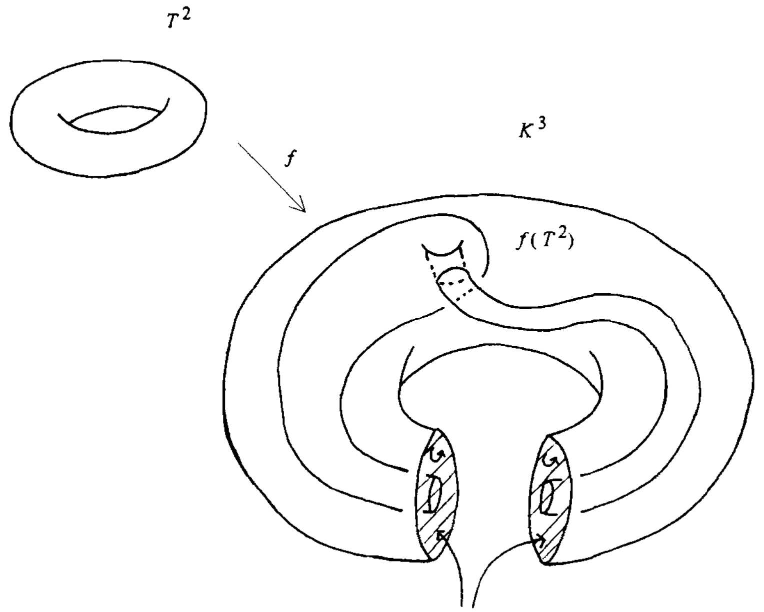
(2) Since $H_{n-1}(N, \partial N) = 0$, by Lemma 2.3(1), we see that $\beta_0(N-f(M)) \geqslant 1 + \beta_{n-1}(f(M), f(\partial M))$ (in fact, this is an equality by Lemma 2.5). On the other hand, since $H_1(M) = 0$, the normal bundle of the immersion f is trivial. Then the same argument as above shows that $\beta_0(N-f(M)) \geqslant 3$.

(3) and (4) The proofs are the same as that of (2). □

Remark 2.8. Consider the immersion with normal crossings $f: T^2 \to K^3$ as in Fig. 2, where T^2 is the torus and K^3 is the open solid Klein bottle (i.e., K is the total space of a nonorientable $\mathbb{R}^2$-bundle over S^1). Note that $H_2(K^3) = 0$, that f is not an embedding and that the number of connected components of $K^3 - f(T^2)$ is equal to 2. This example shows that the condition $H_1(M) = 0$ in Theorem 1.2(2) and the condition that N be orientable in Theorem 1.2(3) are essential. Note that, in this example, the normal bundle of the immersion is not trivial.

Remark 2.9. If both M and N are orientable, Theorem 1.2 is proved in a more general context in [6]. Note that there exist (open) n-manifolds N with $H_{n-1}(N, \partial N) = 0$ which is nonorientable. In this case, the result of [6] cannot be applied.

Proof of Theorem 1.3. Since $H_1(N) = 0$ or $H_{n-1}(N, \partial N) = 0$, by Lemma 2.2 or by Lemma 2.7, we see that $\beta_0(N-f(M)) = 1 + \beta_{n-1}(f(M), f(\partial M))$. Thus, by Lemma 2.3(2), we have $\beta_0(N-f(M)) = 2 + \dim \ker(\alpha: H_{n-2}(A, \partial A) \to H_{n-2}(B, \partial B) \oplus H_{n-2}(M, \partial M))$. On the other hand, when the self-intersection set of f is reduced to the set of double points, we see easily that $\dim \ker((f \mid A)_*: H_{n-2}(A, \partial A) \to H_{n-2}(B, \partial B)) = \beta_0(B)$, since $f \mid A: A \to B$ is a double covering between compact



Attach here by an orientation preserving diffeomorphism.

Fig. 2.

$(n-2)$-manifolds with boundary. Hence we have $\beta_0(N-f(M)) \leqslant 2 + \dim \ker(f \mid A)_* = 2 + \beta_0(B)$. Note that, if $(A, \partial A)$ is null-homologous in $(M, \partial M)$ (i.e., if $j_* : H_{n-2}(A, \partial A) \to H_{n-2}(M, \partial M)$ vanishes), then $\ker \alpha = \ker(f \mid A)_*$. This completes the proof. □

Remark 2.10. When $n = 3$ and M is closed, Theorem 1.3 has been obtained by Izumiya and Marar [4]. Here we note that their assumption that $C(f) = 0$ (i.e., f has no cross-caps) in their Theorem 1.2(2) of [4] is not necessary. The same proof as theirs is valid for this case as well. We also note that, when M is closed, Theorem 1.3 has recently been obtained independently by Nuño Ballesteros [5].

3. Applications

As a consequence of Theorem 1.2, one gets a characterization of embeddings among codimension-1 immersions with normal crossings.

Corollary 3.1. *Let M and N be connected manifolds with boundary of dimensions $n-1$ and n respectively such that M is compact. Suppose that one of the following holds.*

(1) *M is orientable and* $H_1(N; \mathbb{Z}_2) = 0$.
(2) $H_1(M; \mathbb{Z}_2) = 0$ *and* $H_{n-1}(N, \partial N; \mathbb{Z}_2) = 0$.
(3) *M and N are orientable and* $H_{n-1}(N, \partial N; \mathbb{Z}_2) = 0$.

Then an immersion with normal crossings $f: M \to N$ *is an embedding if and only if* $f(M)$ *separates N into exactly two connected components.*

Remark 3.2. If N is closed, Corollary 3.1 has been obtained by Biasi, Motta and Saeki in [2]. Note also that, under the hypothesis that N be closed and $H_1(N) = H_1(M) = 0$, Corollary 3.1 has been obtained by Biasi and Fuster in [1].

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